

A Proof of the Bomber Problem's Spend-It-All Conjecture

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Abstract: The Bomber Problem concerns optimal sequential allocation of partially effective ammunition x while under attack from enemies arriving according to a Poisson process over a time interval of length t . In the doubly-continuous setting, in certain regions of (x, t) -space we are able to solve the integral equation defining the optimal survival probability and find the optimal allocation function $K(x, t)$ exactly in these regions. As a consequence, we complete the proof of the “spend-it-all” conjecture of Bartroff et al. (2010b) which gives the boundary of the region where $K(x, t) = x$.

Keywords: Ammunition rationing; Optimal allocation; Poisson process; Sequential optimization.

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1. INTRODUCTION

The Bomber Problem, first posed by Klinger and Brown (1968), concerns an aircraft equipped with x units of ammunition that is t time units away from its final destination. It is confronted by enemy airplanes whose appearance is driven by a time-homogenous Poisson process with known intensity, taken to be 1. In the encounters, which are assumed to be instantaneous, if the Bomber fires y of its available x units of ammunition, then the probability that the Bomber survives the encounter is given by

$$a(y) = 1 - (1 - u)e^{-y} \quad (1.1)$$

for some fixed $0 \leq u < 1$. This can be interpreted as the Bomber's y units of ammunition destroying the enemy with probability $1 - e^{-y}$, while otherwise allowing the enemy to launch a counterattack which succeeds with probability $1 - u$. The optimal amount of ammunition that should be spent when the Bomber is confronted by an enemy while in “state” (x, t) in order to maximize the probability of reaching its destination is denoted $K(x, t)$, and the optimal probability is denoted $P(x, t)$. Of central interest in the Bomber Problem is a set of conjectures concerning monotonicity of $K(x, t)$; see Bartroff et al. (2010a) and references therein for a description of the conjectures and their statuses.

In this paper we adopt the doubly-continuous setting where x and t are both assumed to be continuous variables, and show in Theorem 2.1 that $K(x, t)$ and $P(x, t)$ can be solved for exactly in certain regions of (x, t) -space. This is done by solving the integral equation

$$P(x, t) = e^{-t} \left(1 + \int_0^t \max_{0 \leq y \leq x} a(y) P(x - y, s) e^s ds \right), \quad (1.2)$$

of which Bartroff et al. (2010a, Corollary 2.2) showed that the the optimal survival probability $P(x, t)$ is the unique solution, and showing that the maximum in the integrand of (1.2) is uniquely achieved, giving

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$K(x, t)$. This allows us in Corollary 2.1 to complete the proof of a conjecture by Bartroff et al. (2010b) that the “spend-it-all” region, i.e., the set of all (x, t) such that $K(x, t) = x$, is given by

$$R_1 = \{(x, t) \in \mathbb{R}^+ \times \mathbb{R}^+ : x \leq f_u(t)\}, \quad (1.3)$$

where

$$f_u(t) = \begin{cases} \log(1 + u/(e^{tu} - 1)), & 0 < u \leq 1 \\ \log(t^{-1} + 1), & u = 0. \end{cases} \quad (1.4)$$

Regarding the Bomber Problem’s monotonicity conjectures we only remark that the form of $K(x, t)$ found in (2.1) is increasing in x , and hence does not violate the conjecture that $K(x, t)$ obeys this everywhere, although this remains unproved in general at this time; see Bartroff et al. (2010a).

2. Results

Theorem 2.1. *Let R_1 be as in (1.3) and*

$$R_2 = \{(x, t) \in \mathbb{R}^+ \times \mathbb{R}^+ : f_u(t) < x \leq 2f_u(t)\}.$$

For all $0 \leq u < 1$,

$$K(x, t) = \begin{cases} x, & (x, t) \in R_1 \\ (x + f_u(t))/2, & (x, t) \in R_2. \end{cases} \quad (2.1)$$

For all $0 < u < 1$,

$$P(x, t) = e^{-t} \times \begin{cases} 1 + a(x)(e^{tu} - 1)/u, & (x, t) \in R_1 \\ 1 + \frac{a(x)}{e^x - 1} + \int_{f_u(x/u)/u}^t u^{-1} (\sqrt{e^{su} - 1} + u - (1 - u)e^{-x/2} \sqrt{e^{su} - 1})^2 ds, & (x, t) \in R_2, \end{cases} \quad (2.2)$$

and (2.2) holds for $u = 0$ by taking the limit of the right-hand-side as $u \rightarrow 0$.

Proof. We show that the right-hand-side of (2.2) satisfies the integral equation (1.2), and then show that the maximum in (1.2) is uniquely achieved at (2.1). Assume that $0 < u < 1$. The proof for $u = 0$ is exactly the same after replacing all quantities involving u by their limit as $u \rightarrow 0$. For example, the integrand in the second case of (2.2) becomes

$$\left(\sqrt{s + 1} - e^{-x/2} \sqrt{s} \right)^2.$$

Letting $\bar{P}(x, t) = e^t P(x, t)$, (1.2) becomes

$$\bar{P}(x, t) = 1 + \int_0^t \max_{0 \leq y \leq x} a(y) \bar{P}(x - y, s) ds. \quad (2.3)$$

A simple but useful fact is that, for fixed $x, t, B > 0$, the function

$$y \mapsto a(y)(1 + Ba(x - y)) \quad \text{is unimodal about} \quad y^* = \frac{x + \log(1 + 1/B)}{2}, \quad (2.4)$$

which can be verified by basic calculus.

Fix $(x, t) \in R_1$ and let $0 \leq s \leq t$ and

$$G_1(y, s) = a(y) [1 + a(x - y)(e^{su} - 1)/u].$$

In order to compute $\max_{0 \leq y \leq x} G_1(y, s)$ we apply (2.4) with $B = (e^{su} - 1)/u$ and, using that f_u is decreasing, we have

$$\left. \frac{x + \log(1 + 1/B)}{2} \right|_{B=(e^{su}-1)/u} = \frac{x + f_u(s)}{2} \geq \frac{x + f_u(t)}{2} \geq \frac{x + x}{2} = x.$$

Thus

$$\max_{0 \leq y \leq x} G_1(y, s) = G_1(x, s) = a(x)e^{su}, \quad (2.5)$$

hence

$$1 + \int_0^t \max_{0 \leq y \leq x} G_1(y, s) ds = 1 + \int_0^t a(x)e^{su} ds = 1 + a(x)(e^{tu} - 1)/u,$$

giving the first case of (2.2), and also (2.1) since the maximum is uniquely achieved at $y = x$.

Now fix $(x, t) \in R_2$. Let $v = 1 - u$ and

$$q(y, s) = \left(\sqrt{e^{su} - v} - ve^{-y/2} \sqrt{e^{su} - 1} \right)^2 / u, \quad (2.6)$$

$$Q_2(y, s) = 1 + \frac{a(y)}{e^y - 1} + \int_{f_u(y/u)/u}^s q(y, r) dr, \quad (2.7)$$

$$Q(y, s) = \begin{cases} 1 + a(y)(e^{su} - 1)/u, & (y, s) \in R_1 \\ Q_2(y, s), & (y, s) \in R_2, \end{cases} \quad (2.8)$$

$$G(s) = \max_{0 \leq y \leq x} a(y)Q(x - y, s). \quad (2.9)$$

Let $0 \leq s \leq t$ and note that $f_u^{-1}(y) = f_u(y/u)/u$. If $s \leq f_u(x/u)/u$, then $(x - y, s) \in R_1$ for all $0 \leq y \leq x$ since $x - y \leq x \leq f_u(s)$, hence

$$G(s) = a(x)e^{su} \quad \text{for } 0 \leq s \leq f_u(x/u)/u \quad (2.10)$$

by (2.5). Now assume that $f_u(x/u)/u \leq s \leq t$. Then

$$(x - y, s) \in \begin{cases} R_1, & x - f_u(s) \leq y \leq x \\ R_2, & 0 \leq y \leq x - f_u(s), \end{cases}$$

hence

$$G(s) = \max \left\{ \max_{0 \leq y \leq x - f_u(s)} a(y)Q_2(x - y, s), \max_{x - f_u(s) \leq y \leq x} G_1(y, s) \right\}. \quad (2.11)$$

To compute the second term in (2.11), we again apply (2.4) with $B = (e^{su} - 1)/u$ but this time note that

$$\left. \frac{x + \log(1 + 1/B)}{2} \right|_{B=(e^{su}-1)/u} = \frac{x + f_u(s)}{2} \in [x - f_u(s), x], \quad (2.12)$$

hence

$$\max_{x - f_u(s) \leq y \leq x} G_1(y, s) = G_1((x + f_u(s))/2, s) = q(x, s), \quad (2.13)$$

after some simplification. Letting $G_2(y, s) = a(y)Q_2(x - y, s)$ and $G'_2(y, s) = (\partial/\partial y)G_2(y, s)$, we show that $G(s)$ is in fact equal to (2.13) by showing that $G'_2(y, s) > 0$ for all $0 \leq y \leq x - f_u(s)$. Letting $Q'_2(y, s) = (\partial/\partial y)Q_2(y, s)$, using the fundamental theorem of calculus we have

$$\begin{aligned} Q'_2(y, s) &= \frac{2v - ve^{-y} - e^y}{(e^y - 1)^2} - \left(\frac{\partial}{\partial y} f_u(y/u)/u \right) q(y, f_u(y/u)/u) + \int_{f_u(y/u)/u}^s \frac{\partial}{\partial y} q(y, r) dr \\ &= \frac{ve^{-y}}{e^y - 1} + \frac{ve^{-y/2}}{u} \int_{f_u(y/u)/u}^s \sqrt{e^{ru} - 1} \left(\sqrt{e^{ru} - v} - ve^{-y/2} \sqrt{e^{ru} - 1} \right) dr. \end{aligned} \quad (2.14)$$

Letting $I_1(y, s)$ be the second term in (2.14) and $I_2(y, s)$ the integral in (2.7), we have

$$\begin{aligned} G'_2(y, s) &= a'(y)Q_2(x - y, s) - a(y)Q'_2(x - y, s) \\ &= ve^{-y} \left(1 + \frac{a(x - y)}{e^{x-y} - 1} + I_2(x - y, s) \right) - a(y) \left(\frac{ve^{-x+y}}{e^{x-y} - 1} + I_1(x - y, s) \right). \end{aligned} \quad (2.15)$$

The terms in (2.15) not involving integrals are

$$ve^{-y} \left(1 + \frac{a(x - y)}{e^{x-y} - 1} \right) - a(y) \left(\frac{ve^{-x+y}}{e^{x-y} - 1} \right) = \frac{v(e^{x-2y} - e^{-x+y})}{e^{x-y} - 1} > 0 \quad (2.16)$$

since, comparing exponents in the numerator and using that $y \leq x - f_u(s)$,

$$\begin{aligned} x - 2y &= 2x - 3y - x + y \\ &\geq 2x - 3(x - f_u(t)) - x + y \\ &= (3f_u(t) - x) - x + y \\ &> -x + y. \end{aligned}$$

The terms in (2.15) involving integrals are

$$\begin{aligned} &ve^{-y}I_2(x - y, s) - a(y)I_1(x - y, s) = \\ &\frac{ve^{-(x-y)/2}}{u} \int_{f_u((x-y)/u)}^s \left(\sqrt{e^{ru} - v} - ve^{-(x-y)/2} \sqrt{e^{ru} - 1} \right) \sqrt{e^{ru} - 1} \left(e^{x/2 - 3y/2 + f_u(r)/2} - 1 \right) dr \geq 0, \end{aligned} \quad (2.17)$$

since the integrand is nonnegative: The first two factors are clearly so and the third factor is as well since, examining the exponent and using that $y \leq x - f_u(s)$ and f_u is decreasing,

$$\begin{aligned} x/2 - 3y/2 + f_u(r)/2 &\geq x/2 - 3(x - f_u(s))/2 + f_u(s)/2 \\ &= -x + 2f_u(s) \\ &\geq -x + 2f_u(t) \\ &\geq 0 \end{aligned}$$

since $(x, t) \in R_2$. Combining (2.16) and (2.17) shows that $G'_2(y, s) > 0$ for all $0 \leq y \leq x - f_u(s)$, and hence that $G(s)$ is equal to (2.13) for $f_u(x/u)/u \leq s \leq t$. Note that in this case the maximum in $G(s)$ is uniquely attained at y equal to (2.12). Then, using (2.10),

$$\begin{aligned} 1 + \int_0^t \max_{0 \leq y \leq x} a(y)Q(x - y, s)ds &= 1 + \int_0^t G(s)ds \\ &= 1 + \int_0^{f_u(x/u)/u} a(x)e^{su} ds + \int_{f_u(x/u)/u}^t q(x, s)ds \\ &= 1 + \frac{a(x)}{e^x - 1} + \int_{f_u(x/u)/u}^t q(x, s)ds \\ &= Q_2(x, t) \\ &= Q(x, t), \end{aligned}$$

this last since $(x, t) \in R_2$, showing that Q satisfies the integral equation in R_2 and hence showing that the second case of (2.1) and (2.2) hold. \square

As a consequence of Theorem 2.1, we complete the proof of the spend-it-all conjecture:

Corollary 2.1. $K(x, t) = x$ if and only if $x \leq f_u(t)$.

Proof. Bartroff et al. (2010b, Theorem 2.1) showed that $K(x, t) < x$ if $x > f_u(t)$, and Theorem 2.1 provides the converse. \square

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