Supplementary Material for "Optimal and fast confidence intervals for hypergeometric successes"

Jay Bartroff*, Gary Lorden†, and Lijia Wang‡

*Department of Statistics and Data Sciences, University of Texas at Austin, Austin, Texas, USA

†Department of Mathematics, Caltech, Pasadena, California, USA

‡Department of Mathematics, University of Southern California, Los Angeles, California, USA

Sections, theorems, and equations in this supplement have prefix S. Those without the prefix refer to the original paper.

S.1 Properties of the hypergeometric distribution and auxiliary lemmas

In this section, we first record some properties of the Hyper(M, n, N) distribution in Proposition S.1.1 and Lemmas S.1.1 and S.1.2, the last covering unimodality of $P_M(x)$ in x. These lemmas are well known and the content of Lemma S.1.1 is mentioned in Johnson et al. (1993, Chapter 6), so we do not prove that here. Lemma S.1.2 follows from the expression

$$\frac{P_M(x)}{P_M(x-1)} = \frac{(M+1-x)(n+1-x)}{x(N-M-n+x)} \quad \text{for} \quad 1 \le x \le n.$$

After that we state and prove some needed auxiliary results concerning other types of monotonicity and unimodality: monotonicity of density function ratios with respect to M in Lemma S.1.3, and unimodality of $P_M([a,b])$ with respect to M in Lemma S.1.4 and with respect to shifts in the interval $[a,b]$ in Lemma S.1.5. Throughout let $P_M(x)$ denote the density (1) of the Hyper(M,n,N) distribution.

Proposition S.1.1. *If S is P_M-maximizing for some M then it is a subinterval of $[x_{\min}, x_{\max}]$, as defined in (2).*

Proof. By definition $S \subseteq [x_{\min}, x_{\max}]$, and by the (near-strict) unimodality of every P_M in Lemma S.1.2 there exists $[m_1, m_2] \subseteq [x_{\min}, x_{\max}]$ with $m_2 - m_1 = 0$ or 1 such that $P_M(x)$ is strictly increasing on $[x_{\min}, m_1]$ and strictly decreasing on $[m_2, x_{\max}]$. Suppose $x_1, x_2 \in S$ and $x_1 < y < x_2$. Then $y \le m_1$ implies $P_M(y) > P_M(x_1)$ and $y \ge m_2$ implies $P_M(y) > P_M(x_2)$. Since m_1 and m_2 are equal or adjacent, at least one of these cases applies, and therefore $P_M(y) > \min\{P_M(x_1), P_M(x_2)\}$. By Lemma S.3.4 it follows that $y \in S$ and thus S is an interval. □

Lemma S.1.1. *1. We have*

$$X \sim \textit{Hyper}(n, M, N) \Leftrightarrow X \sim \textit{Hyper}(M, n, N)$$
$$\Leftrightarrow n - X \sim \textit{Hyper}(N - M, n, N). \qquad \text{(S.1)}$$

2. *A useful coupling: For $n < N$, $X \sim \textit{Hyper}(M, n+1, N)$ can be written $X = X' + Y$ where $X' \sim \textit{Hyper}(M, n, N)$ and $Y|X' \sim \textit{Bern}((M - X')/(N - n))$.*

3. *Monotone likelihood ratio: For every $M_1, M_2 \in [N]$ with $M_1 < M_2$, $P_{M_2}(x)/P_{M_1}(x)$ is nondecreasing in x (with the convention $c/0 = \infty$ for $c > 0$).*

Lemma S.1.2 (Unimodality properties of the hypergeometric)**.** *Let*

$$m = \frac{(n+1)(M+1)}{N+2}, \tag{S.2}$$

$m_1 = \lceil m - 1 \rceil$, *and* $m_2 = \lfloor m \rfloor$. *The following hold.*

1. $P_M(x)$ *increases strictly on* $[x_{\min}, m_1]$ *and decreases strictly on* $[m_2, x_{\max}]$, *where* $x_{\min} = \max\{0, M+n-N\}$ *and* $x_{\max} = \min\{n, M\}$ *are the smallest and largest, respectively, of the* x *values with positive* P_M *probability.*

2. $\arg\max_x P_M(x) = [m_1, m_2]$, *where* $[m_1, m_2] = [\lfloor m \rfloor, \lfloor m \rfloor]$, *unless* m *is an integer, in which case* $[m_1, m_2] = [m-1, m]$.

The next lemma establishes monotonicity in M of ratios $P_M(x_2)/P_M(x_1)$.

Lemma S.1.3. *For fixed* N *and* n, *let* x_1, x_2 *be distinct integers in* $[0, n]$ *such that* $0 < x_2 - x_1 < N - n$. *Then*

$$\frac{P_M(x_2)}{P_M(x_1)} < \frac{P_{M+1}(x_2)}{P_{M+1}(x_1)} \quad \text{for} \quad x_2 \leq M \leq N - n + x_1. \tag{S.3}$$

Proof. We have

$$\frac{P_M(x_2)}{P_M(x_1)} = \prod_{x=x_1}^{x_2-1} \frac{(M-x)(n-x)}{(N-M-(n-x)+1)(x+1)} \tag{S.4}$$

$$< \prod_{x=x_1}^{x_2-1} \frac{(M+1-x)(n-x)}{(N-(M+1)-(n-x)+1)(x+1)} = \frac{P_{M+1}(x_2)}{P_{M+1}(x_1)}. \tag{S.5}$$

□

The next lemma establishes the unimodality of probabilities $P_M([a,b])$ as a function of M. It is helpful to define coupled random variables X and Y as the numbers of red and white balls, respectively, in a simple random sample of n from a box of N balls in which M balls are white, one is red, and the remaining $N-(M+1)$ balls are green. Then $X \sim \text{Hyper}(M, n, N)$ and $X+Y \sim \text{Hyper}(M+1, n, N)$. In the usual notation, $P_{M+1}(x) = P(X+Y=x)$ and $P_M(x) = P(X=x)$. Writing

$$\begin{aligned} P_{M+1}(x) - P_M(x) &= [P(X=x-1, Y=1) + P(X=x, Y=0)] \\ &\quad - [P(X=x, Y=1) + P(X=x, Y=0)] \\ &= P(X=x-1, Y=1) - P(X=x, Y=1), \end{aligned}$$

and summing over x from a to b yields

$$P_{M+1}([a,b]) - P_M([a,b]) = P(X=a-1, Y=1) - P(X=b, Y=1). \tag{S.6}$$

Note that for x such that $P(X=x) > 0$,

$$P(X=x, Y=1) = P(X=x)\frac{n-x}{N-M} = P_M(x)\frac{n-x}{N-M} \tag{S.7}$$

since x white balls in the sample implies that $n-x$ of the $N-M$ colored (red or green) balls are in the sample, so that the red ball has conditional probability $(n-x)/(N-M)$ of being in the sample. Relation (S.7) is trivially true when $P(X=x) = 0$, so it is true for all $x \in [n]$. Using (S.6) and (S.7),

$$(N-M)(P_{M+1}([a,b]) - P_M([a,b])) = (n-(a-1))P_M(a-1) - (n-b)P_M(b). \tag{S.8}$$

This equation provides the basis for the following lemma.

Lemma S.1.4. *Assume $0 \le a \le b \le n$ and $b - a < n$. Then $P_M([a,b])$ is nondecreasing for $M \le M(a,b)$ and nonincreasing for $M \ge M(a,b)$, where*

$$M(a,b) = \begin{cases} 0 & \text{if } a = 0 \\ N & \text{if } b = n \\ \min\{M : (n-(a-1))P_M(a-1) < (n-b)P_M(b)\} & \text{otherwise.} \end{cases} \tag{S.9}$$

Proof. By (S.8),

$$\mathrm{sgn}(P_{M+1}([a,b]) - P_M([a,b])) = \mathrm{sgn}((n-(a-1))P_M(a-1) - (n-b)P_M(b)). \tag{S.10}$$

If $a = 0$, the first term on the right-hand side vanishes and $\{P_M([a,b])\}$ is therefore nonincreasing. Similarly if $b = n$, $\{P_M([a,b])\}$ is nondecreasing. It remains to consider only $1 \le a \le b \le n-1$, and it suffices to show that if M_1, M_2 are such that

$$\mathrm{sgn}(P_{M+1}([a,b]) - P_M([a,b])) = \begin{cases} +1 & \text{if } M = M_1 \\ -1 & \text{if } M = M_2, \end{cases} \tag{S.11}$$

then $M_1 < M_2$. Since the coefficients $n - (a-1)$ and $n - b$ in (S.10) are positive, (S.10) and (S.11) imply that $P_{M_1}(a-1)$ and $P_{M_2}(b)$ must be positive. Therefore, since by (2) $P_M(x)$ must be positive if and only if $x \le M \le x + N - n$, we have

$$M_1 \in I_1 := [a-1, a-1+N-n] \quad \text{and} \quad M_2 \in I_2 := [b, b+N-n].$$

The endpoints of I_1 are less than the corresponding endpoints of I_2, so that M_1 cannot be to the right of I_2, and if it is to the left, $M_1 < M_2$ follows immediately.

So assume that M_1 belongs to I_2 and similarly that M_2 belongs to I_1. Then M_1 and M_2 both belong to $I_1 \cap I_2$, hence

$$b \leq M_1, M_2 \leq a - 1 + N - n.$$

Since $n - b$ is positive and $P_M(a-1)$ is positive on this interval, (S.10) and (S.11) imply that

$$\operatorname{sgn}\left(\frac{P_M(b)}{P_M(a-1)} - \frac{n-(a-1)}{n-b}\right) = \begin{cases} -1 & \text{if } M = M_1 \\ +1 & \text{if } M = M_2, \end{cases}$$

and the property (S.3) implies that $M_1 < M_2$. □

Lemma S.1.5. *For fixed n, N, $0 \leq a \leq b < n$, and positive integer $d \leq n - b$, we have*

(i) $M(a, b) \leq M(a + d, a + d)$, *and*

(ii) $P_M([a, b]) \leq P_M([a + d, b + d])$ *for all* $M \geq M(a + d, b + d)$.

Proof. The lemma can be proved by induction on d as it is straightforward to verify using the definition (S.9) of $M(a, b)$ and the inequality (S.5). We omit the details. □

S.2 Constructing α max optimal intervals: Algorithm S.1

The following is a simple algorithm for producing α max optimal acceptance intervals. The algorithm starts from the endpoints $C = D = \arg\max_x P_M(x)$ equal to the mode, and moves the endpoints outward, incrementing the acceptance probability. An alternative is to begin from $[C, D] = [0, N]$ and move the endpoints inward, decrementing the probability, although this will be slower in most settings.

Algorithm S.1 Given α, n, and N, produce a set of α max optimal acceptance intervals $\{[a_M, b_M] : M \in [N]\}$.

Require: $N \in \mathbb{N}$, $n \leq N$ and $0 < \alpha < 1$
  for $M = 0, ..., \lfloor N/2 \rfloor$ **do**
    $x_{\min} = \max\{0, M + n - N\}$
    $x_{\max} = \min\{n, N\}$
    $C, D = \lfloor \frac{(n+1)(M+1)}{N+2} \rfloor$
    $P = P_M(C)$
    if $C > x_{\min}$ **then** $PC = P_M(C-1)$ **else** $PC = 0$ **end if**
    if $D < x_{\max}$ **then** $PD = P_M(D+1)$ **else** $PD = 0$ **end if**
    while $P < 1 - \alpha$ **do**
      if $PD > PC$ **then**
        $D = D + 1$
        $P = P + PD$
        if $D < x_{\max}$ **then** $PD = P_M(D+1)$ **else** $PD = 0$ **end if**
      else
        $C = C - 1$
        $P = P + PC$
        if $C > x_{\min}$ **then** $PC = P_M(C-1)$ **else** $PC = 0$ **end if**
      end if
    end while
    $a_M = C$
    $b_M = D$
    $a_{N-M} = n - b_M$
    $b_{N-M} = n - a_M$
  end for
  return $\{[a_M, b_M]\}_{M=0}^{N}$

Lemma S.2.1. *The acceptance intervals* $\{[a_M, b_M] : M \in [\lfloor N/2 \rfloor]\}$ *produced by Algorithm S.1 are* α *max optimal.*

Proof. Fix $M \in [\lfloor N/2 \rfloor]$ and use P to denote P_M. Recall that $P(x) = 0$ for x outside $[x_{\min}, x_{\max}]$. Algorithm S.1 builds up a nested sequence of acceptance intervals $I, J, K, \ldots$ by selecting at each stage a new point having maximum probability among the ones available. The first interval is $I = \{\lfloor m \rfloor\}$ where m is given by (S.2) and $\lfloor m \rfloor$, and thus I, have maximum probability; see Lemma S.1.2. At each successive stage, the current interval J is expanded to $K = J \cup \{x\}$, where x is y, the point adjacent to J on the left, or z, adjacent on the right, chosen to yield the maximum of their probabilities. Since y is to the left of $\{\lfloor m \rfloor\}$, the points further to the left have smaller probabilities by virtue of the unimodality property in Lemma S.1.2, and similarly z has the maximum probability to the right. Therefore the selected point x has the maximum probability outside of the current interval J, and hence by the Definition 2.1 of P-maximizing sets, the next interval K preserves the P-maximizing property of J, using an obvious induction argument.

For the given M, the while loop in Algorithm S.1 ends with the first interval V that is level α. Hence its predecessor is not level α, and since it is P-maximizing, no set with fewer points than V is level α. Thus V is α optimal as well as P-maximizing, and hence it is α max optimal. □

S.3 Proof of Theorem 2.1 and auxiliary results

Proof of Theorem 2.1. We first prove that $\mathcal{M}_a$ and $\mathcal{M}_b$ are disjoint. Take $M \in \mathcal{M}_a$, so that $a_M < \overline{a}_M$, and we will show that $M \notin \mathcal{M}_b$, i.e., $b_M \leq b_{M'}$ for all $M' > M$. Fix such an $M' \in \mathcal{M}$, and we will consider 2 cases, comparing $a_{M'}$ and $\overline{a}_M$. <u>Case 1</u>: $a_{M'} < \overline{a}_M$. In this case we have $a_{M'} < \overline{a}_M \leq \overline{a}_{M'}$. If equality holds in this last, then there exists $M'_\ell < M < M'$ such that $a_{M'_\ell} = \overline{a}_M = \overline{a}_{M'}$, so $b_{M'} \geq b_M$ by part 1(d)ii of Lemma S.3.1. Otherwise, $\overline{a}_M < \overline{a}_{M'}$, so it must be that a new maximum is achieved between M and M', i.e., $\overline{a}_{M'} = a_{M'_\ell}$ for some $M < M'_\ell < M'$. Then $a_{M'_\ell} > \overline{a}_M$, so $b_M < b_{M'_\ell}$ by part 1 of Lemma S.3.2. We also have $a_{M'_\ell} = \overline{a}_{M'} > a_{M'}$, so $b_{M'_\ell} \leq b_{M'}$ by part (i) of Lemma S.3.5. Combining these inequalities gives $b_{M'} > b_M$. Case 2: $a_{M'} \geq \overline{a}_M$. We have $b_{M'} > b_M$ by part 1 of Lemma S.3.2, implying that $b_M = \underline{b}_M$, satisfying the claim.

The proof of that $M \in \mathcal{M}_b$ implies $M \notin \mathcal{M}_a$ is similar.

For monotonicity of the endpoints it suffices to show that $b_M^{adj} \leq b_{M+1}^{adj}$ for $M \leq N-1$, since the corresponding result for the sequence $\{a_M^{adj}\}$ is proved similarly. For all $M \leq N-1$, $b_{M+1}^{adj} \geq \underline{b}_{M+1} \geq \underline{b}_M$, and if $M \notin \mathcal{M}_a$ then $\underline{b}_M = b_M^{adj}$. Hence only the case $M \in \mathcal{M}_a$ remains to be considered. If also $M+1 \in \mathcal{M}_a$, then Lemma S.3.1, part 1d, applies with $M^* = M+1$, hence $b_M - a_M \leq b_{M+1} - a_{M+1}$, and since $\overline{a}_M \leq \overline{a}_{M+1}$, the result follows by adding the last two inequalities. If $M+1 \notin \mathcal{M}_a \cup \mathcal{M}_b$, then $a_{M+1} = \overline{a}_{M+1} \geq \overline{a}_M$, and Lemma S.3.2 applies, yielding $b_{M+1} \geq b_M + \overline{a}_M - a_M$, which suffices. Finally, if $M+1 \in \mathcal{M}_b$, then $M+1 < N$ and

$$b_{M+1} > b_{M+1}^{adj} = \underline{b}_{M+1} = b_{M'} \quad \text{for some} \quad M' > M+1.$$

By Lemma S.3.5, $a_{M'} \geq a_{M+1} = \overline{a}_{M+1} \geq \overline{a}_M$, the equality holding since $M+1 \notin \mathcal{M}_a$ by the disjointness of $\mathcal{M}_a$ and $\mathcal{M}_b$. Applying Lemma S.3.2 and the definition of M', $b_M^{adj} \leq B_{M'} = \underline{b}_{M+1} = b_{M+1}^{adj}$.

That the adjusted intervals are level-α is handled in parts 1a and 2a of Lemma S.3.1, respectively, for the two nontrivial cases. Finally, note that the adjusted intervals have the same length as the original intervals, thus implying length optimality. □

The next lemma establishes that anywhere a "gap" $a_M > \overline{a}_M$ occurs in the sequence of lower endpoints of α max optimal acceptance intervals, the gap may be "filled" by shifting the interval up the needed amount while maintaining the interval's acceptance probability and without violating monotonicity in the upper endpoint b_M.

Lemma S.3.1. *Let $\{[a_M, b_M] : M \in \mathcal{M}\}$ be α max optimal with $\mathcal{M} \subseteq [N]$ an interval, and $\overline{a}_M, \underline{b}_M, \mathcal{M}_a, \mathcal{M}_b$ as defined in (3)-(4).*

1. *If $M^* \in \mathcal{M}_a$ then, letting $\delta = \overline{a}_{M^*} - a_{M^*}$, we have*
   (a) *$P_{M^*}([a_{M^*} + \delta, b_{M^*} + \delta]) \geq 1 - \alpha$,*
   (b) *there exists $M_\ell \in \mathcal{M}$ with $M_\ell < M^*$ such that $a_{M_\ell} = \overline{a}_{M^*}$,*
   (c) *$b_{M^*} + \delta > b_{M_\ell}$ for any M_ℓ satisfying 1b,*
   (d) *for any M_ℓ satisfying 1b, then for all $M \in [M_\ell, M^*]$ we have*
      i. *$b_M - a_M \leq b_{M^*} - a_{M^*}$,*
      ii. *$b_M \leq b_{M^*}$.*
2. *If $M^* \in \mathcal{M}_b$ then, letting $\delta = b_{M^*} - \underline{b}_{M^*}$, we have*
   (a) *$P_{M^*}([a_{M^*} - \delta, b_{M^*} - \delta]) \geq 1 - \alpha$,*
   (b) *there exists $M_u \in \mathcal{M}$ with $M_u > M^*$ such that $b_{M_u} = \underline{b}_{M_u}$,*
   (c) *$a_{M^*} - \delta < a_{M_u}$ for any M_u satisfying 2b,*
   (d) *for any M_u satisfying 2b, then for all $M^* \leq M \leq M_u$ we have*
      i. *$b_M - a_M \leq b_{M^*} - a_{M^*}$,*
      ii. *$a_M \geq a_{M^*}$.*

Proof of Lemma S.3.1. Part 1b follows from the definition of $\overline{a}_M$. For part 1c, $b_{M^*} \geq b_{M_\ell}$ by Lemma S.3.5, so $b_{M^*} + \delta > b_{M_\ell}$. Using this and the fact that

$a_{M^*} + \delta = \overline{a}_{M^*} = a_{M_\ell}$, we have

$$[a_{M_\ell}, b_{M_\ell}] \subseteq [a_{M^*} + \Delta, b_{M^*} + \Delta] \quad \text{for all} \quad \Delta \in [\delta],$$

and thus

$$P_M([a_{M_\ell}, b_{M_\ell}]) \leq P_M([a_{M^*} + \Delta, b_{M^*} + \Delta]) \quad \text{for all} \quad M \in \mathcal{M}, \; \Delta \in [\delta]. \quad \text{(S.12)}$$

We know that

$$P_{M^*}\left([a_{M^*} + \delta, b_{M^*} + \delta]\right) \leq P_{M^*}\left([a_{M^*}, b_{M^*}]\right) \quad \text{(S.13)}$$

by Definition 2.1 since these intervals have the same width. If equality holds in (S.13) then part 1a is proved because the right-hand-side is $\geq 1 - \alpha$. Otherwise strict inequality holds in (S.13), which implies that $M^* < M(a_{M^*} + \delta, b_{M^*} + \delta)$ by Lemma S.1.5, part (ii). Then, using unimodality and (S.12), we have

$$\begin{aligned} P_{M^*}\left([a_{M^*} + \delta, b_{M^*} + \delta]\right) &\geq P_{M_\ell}\left([a_{M^*} + \delta, b_{M^*} + \delta]\right) \\ &\geq P_{M_\ell}\left([a_{M_\ell}, b_{M_\ell}]\right) \\ &\geq 1 - \alpha, \end{aligned}$$

finishing the proof of part 1a. For part 1d, using unimodality we have

$$\begin{aligned} &P_{M^*}([a_{M^*} + \delta, b_{M^*} + \delta]) \geq \\ &\qquad \min(P_{M^*}\left([a_{M^*} + \delta, b_{M^*} + \delta]\right), P_{M_\ell}\left([a_{M^*} + \delta, b_{M^*} + \delta]\right)) \\ &\qquad\qquad \geq 1 - \alpha, \end{aligned}$$

and therefore

$$b_M - a_M \le b_{M^*} + \delta - (a_{M^*} + \delta) = b_{M^*} - a_{M^*}$$

by length optimality of $[a_M, b_M]$. By this inequality, if $a_{M^*} \ge a_M$ then $b_{M^*} \ge b_M$. Otherwise, $a_{M^*} < a_M$ so $b_{M^*} \ge b_M$ by Lemma S.3.5, completing the proof of 1d.

The proof of part 2 involves similar arguments, after reflecting the endpoint sequences $\widetilde{a}_{N-M} = n - b_M$, $\widetilde{b}_{N-M} = n - a_M$ and using Lemma S.3.3. We omit the rest of the details. □

Parts 1-2 of next lemma establish that the adjusted acceptance intervals given in Theorem 2.1 have nondecreasing lower endpoints, and parts 3-4 show the same for the upper endpoints.

Lemma S.3.2. *Let $\{[a_M, b_M] : M \in \mathcal{M}\}$ be α max optimal with $\mathcal{M} \subseteq [N]$ an interval, and $\overline{a}_M, \underline{b}_M, \mathcal{M}_a, \mathcal{M}_b$ as defined in (3)-(4).*

1. *If $M^* \in \mathcal{M}_a$ and $M^* < M \in \mathcal{M}$ satisfy $a_M \ge \overline{a}_{M^*}$, then $b_{M^*} < b_{M^*} + \overline{a}_{M^*} - a_{M^*} \le b_M$.*
2. *The sequence $b_M + \overline{a}_M - a_M$ is nondecreasing in $M \in \mathcal{M}_a$.*
3. *If $M^* \in \mathcal{M}_b$ and $M^* > M \in \mathcal{M}$ satisfy $b_M \le \underline{b}_{M^*}$, then $a_{M^*} > a_{M^*} - (b_{M^*} - \underline{b}_{M^*}) \ge a_M$.*
4. *The sequence $a_M - b_M + \underline{b}_M$ is nondecreasing in $M \in \mathcal{M}_b$.*

Proof. For part 1, there must be $M_\ell < M^*$ such that $a_{M_\ell} = \overline{a}_{M^*} > a_{M^*}$. Then we have $b_{M_\ell} \le b_{M^*}$ by Lemma S.3.5, part (i). Combining these two, we have $[a_{M_\ell}, b_{M_\ell}] \subsetneq [a_{M^*}, b_{M^*}]$, thus

$$P_{M^*}([a_{M_\ell}, b_{M_\ell}]) < 1 - \alpha \le P_{M_\ell}([a_{M_\ell}, b_{M_\ell}]),$$

by length optimality of the latter. This implies that $M^* \geq M(a_{M_\ell}, b_{M_\ell})$ by Lemma S.1.4. We also have that

$$P_M([a_{M_\ell}, b_{M_\ell}]) \leq P_{M^*}([a_{M_\ell}, b_{M_\ell}]) < 1 - \alpha \tag{S.14}$$

since $M > M^* \geq M(a_{M_\ell}, b_{M_\ell})$. If it were that $b_M \leq b_{M_\ell}$, then we would have $[a_M, b_M] \subseteq [a_{M_\ell}, b_{M_\ell}]$ and then (S.14) would imply that

$$P_M([a_M, b_M]) \leq P_M\left([a_{M_\ell}, b_{M_\ell}]\right) < 1 - \alpha,$$

a contradiction. Thus it must be that $b_M > b_{M_\ell}$. Then we have $[a_{M_\ell}, b_{M_\ell}] \subseteq [a_{M_\ell}, b_M]$ and $[a_M, b_M] \subseteq [a_{M_\ell}, b_M]$,

$$P_{M_\ell}([a_{M_\ell}, b_M]) \geq P_{M_\ell}([a_{M_\ell}, b_{M_\ell}]) \geq 1 - \alpha$$

and

$$P_M([a_{M_\ell}, b_M]) \geq P_M([a_M, b_M]) \geq 1 - \alpha\,.$$

Thus, by unimodality, $P_{M^*}([a_{M_\ell}, b_M]) \geq 1 - \alpha$, and so by length optimality we have

$$b_{M^*} - a_{M^*} \leq b_M - a_{M_\ell}. \tag{S.15}$$

Note that $b_{M^*} + \overline{a}_{M^*} - a_{M^*} = a_{M_\ell} + b_{M^*} - a_{M^*}$. By the inequality (S.15), we have

$$b_{M^*} + \overline{a}_{M^*} - a_{M^*} = a_{M_\ell} + b_{M^*} - a_{M^*} \leq a_{M_\ell} + b_M - a_{M_\ell} = b_M,$$

concluding the proof of part 1.

For 2, consider $M_1, M_2 \in \mathcal{M}_a$ with $M_1 < M_2$. We have $\overline{a}_{M_1} \leq \overline{a}_{M_2}$. If strict inequality holds then there is $M_{\ell,2} \in \mathcal{M}$ satisfying that $M_1 < M_{\ell,2} < M_2$ and $a_{M_{\ell,2}} = \overline{a}_{M_2} > \overline{a}_{M_1}$. Then using part 1 and Lemma S.3.1, respectively, for the following inequalities,

$$b_{M_1} + \overline{a}_{M_1} - a_{M_1} \leq b_{M_{\ell,2}} < b_{M_2} + \overline{a}_{M_2} - a_{M_2}.$$

Otherwise $\overline{a}_{M_1} = \overline{a}_{M_2}$ whence there is $M_{\ell,1} \in \mathcal{M}$ with $M_{\ell,1} < M_1$ and $a_{M_{\ell,1}} = \overline{a}_{M_1} = \overline{a}_{M_2}$, thus $b_{M_1} - a_{M_1} \leq b_{M_2} - a_{M_2}$ by Lemma S.3.1. This establishes part 2.

The proof of parts 3-4 involves similar arguments, after reflecting the endpoint sequences $\widetilde{a}_{N-M} = n - b_M$, $\widetilde{b}_{N-M} = n - a_M$ and using Lemma S.3.3. We omit the rest of the details. □

The next lemma shows that α max optimal intervals for $\mathcal{M}$ can be reflected across $N/2$ to produce α max optimal intervals in the reflected set for $\mathcal{M}' = \{N - M : M \in \mathcal{M}\}$.

Lemma S.3.3. *If $\{[a_M, b_M] : M \in \mathcal{M}\}$ are α max optimal for $\mathcal{M}$, then $\{[\widetilde{a}_M := n - b_{N-M}, \widetilde{b}_M := n - a_{N-M}] : M \in \widetilde{\mathcal{M}}\}$ are α max optimal for $\widetilde{\mathcal{M}} := \{N - M : M \in \mathcal{M}\}$.*

Proof of Lemma S.3.3. Fix $M \in \mathcal{M}$ and omit it from the notation. Let $X \sim \text{Hyper}(M, n, N)$ so that $\widetilde{X} := n - X \sim \text{Hyper}(N - M, n, N)$ by (S.1). The three properties in Definition 2.1 are straightforward to verify using that $P(\widetilde{X} \in [a, b]) = P(X \in [n - b, n - a])$. We omit the details. □

The next lemma establishes the point masses inside a probability-maximizing (e.g., α max optimal) set have probabilities no less than those outside, and is used to prove that monotonicity can only be violated one endpoint at a time in α max optimal acceptance intervals, and that probability maximizing sets must be intervals.

Lemma S.3.4. *A set $S \subseteq [x_{\min}, x_{\max}]$ is P_M-maximizing if and only if $x \in S$, $y \notin S$ implies that $P_M(x) \geq P_M(y)$.*

Proof of Lemma S.3.4. The condition is obviously necessary since otherwise replacing x in S by y would increase $P_M(S)$. The converse follows by summing over $x \in S \setminus S^*$ vs. $y \in S^* \setminus S$ for $|S^*| = |S|$. □

The following lemma establishes that, in α max optimal intervals, monotonicity can only be violated one endpoint at a time. This property is used to prove that the sets $\mathcal{M}_a$ and $\mathcal{M}_b$ in Theorem 2.1 are disjoint.

Lemma S.3.5. *Suppose $[a, b]$ and $[a', b']$ are P_M and $P_{M'}$ maximizing, respectively, and $M' > M$.*

(i) If $a' < a$, then $b' \geq b$.

(ii) If $b' < b$, then $a' \geq a$.

(iii) $a' \geq a$ or $b' \geq b$.

Proof of Lemma S.3.5. It suffices to prove part (i) since (i), (ii), and (iii) are logically equivalent.

Assume $a' < a$. Then a' is outside $[a, b]$, and hence $P_M(a') \leq P_M(b)$ by Lemma S.3.4. Then $P_{M'}(a') < P_{M'}(b)$ by Lemma S.1.3, the hypotheses of which follow from (2) and the fact that the intervals are probability maximizing, hence its points have positive probability. Since $[a', b']$ is $P_{M'}$ maximizing, it must contain b. □

S.4 Proofs and auxiliary results for Section 3

Proof of Theorem 3.1. Symmetry is by construction, and monotonicity is proved in Lemma S.4.1. Theorem 2.1 establishes α-optimality in the first case of (10), and to establish it in the second case, let $M > \lfloor N/2 \rfloor$ and $X \sim \text{Hyper}(M, n, N)$. Then

$$P_M(X \in [a^*_M, b^*_M]) = P_M(X \in [n - b^*_{N-M}, n - a^*_{N-M}])$$
$$= P_M(n - X \in [a^*_{N-M}, b^*_{N-M}]) \geq 1 - \alpha,$$

this last by the first case of (10) and since $n - X \sim \text{Hyper}(N - M, n, N)$; see Lemma S.1.1. For size optimality, we will show that $P_M([c, d]) \geq 1 - \alpha$ implies that $d - c \geq b^*_M - a^*_M$. By an argument similar to the one above, $[n - d, n - c]$ is level-α for testing $H(N - M)$ and thus no shorter than $[a^*_{N-M}, b^*_{N-M}]$, so

$$d - c = (n - c) - (n - d) \geq b^*_{N-M} - a^*_{N-M} = (n - a^*_M) - (n - b^*_M) = b^*_M - a^*_M,$$

as claimed. For N even, the third case of (10) is clearly level-α, and any competing symmetrical interval for $H(N/2)$ must be of the form $[c, n - c]$. If this is level-α then $c \leq h_{\alpha/2}$, thus it can be no shorter than (9). □

Lemma S.4.1. *Under the assumptions of Theorem 3.1, the intervals $\{[a^*_M, b^*_M] : M \in [N]\}$ defined by (10) have nondecreasing endpoint sequences.*

Proof. Monotonicity holds separately for M in the first and second cases of (10) by Theorem 2.1. We must show monotonicity "across" $N/2$, i.e.,

$$a^*_{N/2-1} \le a^*_{N/2} \le a^*_{N/2+1} \quad \text{and} \quad b^*_{N/2-1} \le b^*_{N/2} \le b^*_{N/2+1} \quad \text{if } N \text{ is even,} \tag{S.16}$$

and

$$a^*_{\lfloor N/2 \rfloor} \le a^*_{\lfloor N/2 \rfloor+1} \quad \text{and} \quad b^*_{\lfloor N/2 \rfloor} \le b^*_{\lfloor N/2 \rfloor+1} \quad \text{if } N \text{ is odd.} \tag{S.17}$$

For (S.16) take N even, and both sets of inequalities in (S.16) are equivalent to

$$a^{adj}_{N/2-1} \le h_{\alpha/2} \le n - b^{adj}_{N/2-1}. \tag{S.18}$$

By Lemma S.4.2 we have $a_{N/2} \ge h_{\alpha/2} \ge a_M$ for $M \in [N/2-1]$, so $a_{N/2} = \overline{a}_{N/2}$, i.e., $N/2 \notin \mathcal{M}_a$. Note that, by virtue of (3), the upper endpoint b_M for the largest M in the index set never gets adjusted. Here the index set is $[N/2]$ so $\underline{b}_{N/2} = b_{N/2}$, thus $N/2 \notin \mathcal{M}_b$. Therefore $[a_{N/2}, b_{N/2}] = [a^{adj}_{N/2}, b^{adj}_{N/2}]$. By Lemma S.4.2 we also have that $b_{N/2} \le n - h_{\alpha/2}$, and combining these last two gives

$$h_{\alpha/2} \le n - b_{N/2} = n - b^{adj}_{N/2} \le n - b^{adj}_{N/2-1},$$

giving the second inequality in (S.18).

For the first inequality in (S.18), if $a_{N/2-1} \ge a^{adj}_{N/2-1}$ then $a^{adj}_{N/2-1} \le a_{N/2-1} \le h_{\alpha/2}$, using Lemma S.4.2 for this last inequality. Otherwise, $a_{N/2-1} < a^{adj}_{N/2-1}$, meaning $N/2-1 \in \mathcal{M}_a$ so, in particular, there exists $M^* < N/2-1$ such that $a_{M^*} = \overline{a}_{N/2-1} = a^{adj}_{N/2-1}$. Then $a^{adj}_{N/2-1} = a_{M^*} \le h_{\alpha/2}$, using Lemma S.4.2 for the inequality.

To prove (S.17) take N odd and let $M^* = \lfloor N/2 \rfloor$. Both the inequalities in

(S.17) are equivalent to

$$a^{adj}_{M^*} + b^{adj}_{M^*} \leq n.$$

If no adjustment is applied to $[a_{M^*}, b_{M^*}]$, i.e., $[a_{M^*}, b_{M^*}] = [a^{adj}_{M^*}, b^{adj}_{M^*}]$, then this holds by Lemma S.4.3. Otherwise M^* is in $\mathcal{M}_a$ or $\mathcal{M}_b$. If the latter then $[a^{adj}_{M^*}, b^{adj}_{M^*}]$ is shifted down from $[a_{M^*}, b_{M^*}]$, i.e., $a^{adj}_{M^*} \leq a_{M^*}$ and $b^{adj}_{M^*} \leq b_{M^*}$, thus

$$a^{adj}_{M^*} + b^{adj}_{M^*} \leq a_{M^*} + b_{M^*} \leq n$$

using Lemma S.4.3.

Otherwise $M^* \in \mathcal{M}_a$ meaning there is $M' < M^*$ such that $a_{M'} = \overline{a}_{M^*} = a^{adj}_{M^*}$. By Lemma S.4.3 we know that

$$a_{M'} + b_{M'} \leq n. \tag{S.19}$$

Define

$$a'_M = \begin{cases} a_M, & \text{for } M \in [M^*] \\ n - b_{N-M}, & \text{for } M^* < M \leq N, \end{cases} \qquad b'_M = \begin{cases} b_M, & \text{for } M \in [M^*] \\ n - a_{N-M}, & \text{for } M^* < M \leq N. \end{cases}$$

Then $\{[a'_M, b'_M] : M \in [N]\}$ are α max optimal since the $[a_M, b_M]$ are. We now apply part 1 of Lemma S.3.2 to this set, with $M := N - M' > M^*$. We have

$$a'_M = n - b_{M'} \geq a_{M'} = \overline{a}_{M^*} = \overline{a}'_{M^*},$$

where the inequality is by (S.19). Then using Lemma S.3.2 for the following inequality,

$$b_{M^*}^{adj} = b_{M^*} + \overline{a}_{M^*} - a_{M^*} = b'_{M^*} + \overline{a}'_{M^*} - a'_{M^*} \le b'_M = n - a_{M'} = n - a_{M^*}^{adj},$$

the desired inequality. □

Lemma S.4.2. *If N is even and $\{[a_M, b_M] : M \in [N/2]\}$ is any α max optimal set, then*

(i) $[a_{N/2}, b_{N/2}] \subseteq [h_{\alpha/2}, n - h_{\alpha/2}]$,

(ii) for all $M \in [N/2 - 1]$ we have $P_{N/2}([a_M, n - a_M]) \ge 1 - \alpha$ and $a_M \le h_{\alpha/2}$.

Proof. For part (i), toward contradiction suppose that $a_{N/2} < h_{\alpha/2}$. Since $[a_{N/2}, b_{N/2}]$ is length-minimizing, it must also be that $b_{N/2} < n - h_{\alpha/2}$, hence $n - h_{\alpha/2} \notin [a_{N/2}, b_{N/2}]$. Then by Lemma S.3.4,

$$P_{N/2}(a_{N/2}) \ge P_{N/2}(n - h_{\alpha/2}) = P_{N/2}(h_{\alpha/2}), \tag{S.20}$$

this last by symmetry. On the other hand, recall that $h_{\alpha/2} \le \lfloor n/2 \rfloor$; see (11). The mode (S.2) in this case is $m = (n + 1)/2$. We have

$$a_{N/2} < h_{\alpha/2} \le \lfloor n/2 \rfloor \le \begin{cases} \lfloor m \rfloor - 1, & \text{if } n \text{ is odd } (m = \lfloor m \rfloor), \\ \lfloor m \rfloor & \text{if } n \text{ is even } (m \ne \lfloor m \rfloor). \end{cases}$$

By this and Lemma S.1.2 we have that $P_{N/2}(h_{\alpha/2}) \ge P_{N/2}(a_{N/2})$. Moreover, this inequality is strict by part 1 of Lemma S.1.2 since the latter is positive, $a_{N/2}$ being an endpoint of an α max optimal interval, hence the former is positive too. The strict inequality contradicts (S.20).

If $b_{N/2} > n - h_{\alpha/2}$ then similar arguments apply.

For part (ii), for $M \in [N/2-1]$, using Lemma S.4.3 to show inclusion of the following intervals, we have

$$P_M([a_M, n-a_M]) \geq P_M([a_M, b_M]) \geq 1-\alpha, \quad \text{and}$$
$$P_{N-M}([a_M, n-a_M]) \geq P_{N-M}([n-b_M, n-a_M]) \geq 1-\alpha.$$

The first claim of part (ii) then follows from these inequalities and Lemma S.1.4, which says that $M' \mapsto P_{M'}([a_M, n-a_M])$ is unimodal, thus $P_{N/2}([a_M, n-a_M]) \geq 1-\alpha$. This inequality is equivalent to

$$\alpha \geq P_{N/2}(X < a_M) + P_{N/2}(X > n-a_M) = 2P_{N/2}(X < a_M),$$

by symmetry, and thus $a_M \leq h_{\alpha/2}$ by definition of the latter, establishing the second claim. □

Lemma S.4.3. *Let* $\{[a_M, b_M] \mid M = [\lfloor N/2 \rfloor]\}$ *be* α *max optimal. Then*

$$a_M + b_M \leq n \quad \textit{for all} \quad M < N/2. \tag{S.21}$$

Proof. Suppose this fails, so that $b_{M^*} > n - a_{M^*}$ for some $M^* \in \mathcal{M} := [\lfloor (N-1)/2 \rfloor]$. By Lemma S.3.3,

$$\{[\widetilde{a}_M = n - b_{N-M}, \widetilde{b}_M = n - a_{N-M}] : \; M \in \widetilde{\mathcal{M}} := N - \mathcal{M}\} \tag{S.22}$$

is α max optimal for $\widetilde{\mathcal{M}}$. Then, since

$$b_{M^*} \notin [n - b_{M^*}, n - a_{M^*}] = [\widetilde{a}_{N-M^*}, \widetilde{b}_{N-M^*}],$$

by Lemma S.3.4 we have that

$$P_{N-M^*}(b_{M^*}) \leq P_{N-M^*}(n - b_{M^*}). \tag{S.23}$$

By similar arguments, since $n - b_{M^*} \notin [a_{M^*}, b_{M^*}]$ we have that $P_{M^*}(b_{M^*}) \geq P_{M^*}(n - b_{M^*})$. Next we will apply Lemma S.1.3 with $x_1 = b_{M^*}$ and $x_2 = n - b_{M^*}$. We have

$$b_{M^*} \leq M^* < N/2 < N - M^* \leq N - n + n - b_{M^*},$$

so that lemma tells us that

$$\frac{P_{N-M^*}(b_{M^*})}{P_{N-M^*}(n - b_{M^*})} > \frac{P_{M^*}(b_{M^*})}{P_{M^*}(n - b_{M^*})} \geq 1,$$

which contradicts (S.23) and thus establishes (S.21). □

S.5 Algorithm S.2

Algorithm S.2 Given α, n, and N, calculate a set of level-α acceptance intervals $\{[a^*_M, b^*_M] : M \in [N]\}$.

Require: $N \in \mathbb{N}$, $n \leq N$ and $0 < \alpha < 1$

```
for M = 0, ..., ⌊N/2⌋ do
    x_min = max{0, M + n − N}
    x_max = min{n, N}
    C, D = ⌊(n+1)(M+1)/(N+2)⌋
    P = P_M(C)
    if C > x_min then PC = P_M(C − 1) else PC = 0 end if
    if D < x_max then PD = P_M(D + 1) else PD = 0 end if
    while P < 1 − α do
        if PD > PC then
            D = D + 1, P = P + PD
            if D < x_max then PD = P_M(D + 1) else PD = 0 end if
        else
            C = C − 1, P = P + PC
            if C > x_min then PC = P_M(C − 1) else PC = 0 end if
        end if
    end while
    a_M = C, b_M = D
end for
b*_0 = b_0, a*_0 = a_0
for M = 1, . . . , ⌊N/2⌋ do
    if a_M < a*_{M−1} then
        a*_M = a*_{M−1}, b*_M = b_M + a*_{M−1} − a_M
    else
        b*_M = b_M, a*_M = a_M
    end if
end for
for M = ⌊N/2⌋ - 1,. . . , 0 do
    if b*_M > b*_{M+1} then
        a*_M = a*_M + b*_{M+1} − b*_M, b*_M = b*_{M+1}
    end if
    a*_{N−M} = n − b*_M, b*_{N−M} = n − a*_M
end for
if N is even then
    a*_{N/2} = max{a_{N/2}, n − b_{N/2}}, b*_{N/2} = n − a*_{N/2}
else
    a*_{⌊N/2⌋+1} = n − b*_{⌊N/2⌋}, b*_{⌊N/2⌋+1} = n − a*_{⌊N/2⌋}
end if
return {[a*_M, b*_M]}_{M=0}^{N}
```

S.6 Proofs and auxiliary results for Section 4

Proof of Lemma 4.1. Denote $\mathcal{C}_{\mathcal{A}}$ simply by $\mathcal{C}$. For part 1, letting $\mathbf{1}\{\cdot\}$ denote the indicator function,

$$
\begin{aligned}
|\mathcal{C}| = \sum_{x\in[n]} |\mathcal{C}(x)| = \sum_{x\in[n]} |\{M \in [N] : x \in \mathcal{A}(M)\}| \\
= \sum_{x\in[n],\, M\in[N]} \mathbf{1}\{x \in \mathcal{A}(M)\} = \sum_{M\in[N]} |\{x \in [n] : x \in \mathcal{A}(M)\}| \\
= \sum_{M\in[N]} |\mathcal{A}(M)| = |\mathcal{A}|.
\end{aligned}
$$

For part 2, if $\mathcal{A}$ is symmetrical,

$$
\begin{aligned}
\mathcal{C}(x) &= \{M \in [N] : \ x \in \mathcal{A}(M)\} \\
&= \{M \in [N] : \ x \in n - \mathcal{A}(N-M)\} \\
&= \{M \in [N] : \ n - x \in \mathcal{A}(N-M)\} \\
&= \{N - M \in [N] : \ n - x \in \mathcal{A}(M)\} \\
&= N - \{M \in [N] : \ n - x \in \mathcal{A}(M)\} \\
&= N - \mathcal{C}(n-x).
\end{aligned}
$$

A similar argument shows the converse.

For part 3, fix arbitrary $x \in [n]$ and to show that $\mathcal{C}(x)$ is an interval, suppose that $M_1, M_2 \in \mathcal{C}(x)$ and we will show that $M \in \mathcal{C}(X)$ for all $M_1 < M < M_2$. Since $M_2 \in \mathcal{C}(x)$, $x \in [a_{M_2}, b_{M_2}]$ so $x \geq a_{M_2} \geq a_M$ by monotonicity. By a similar argument, $x \leq b_{M_1} \leq b_M$, thus $x \in [a_M, b_M]$ so $M \in \mathcal{C}(x)$.

□

Lemma S.6.1. *In the setting of Theorem 4.1, for any $\mathcal{C} \in \mathfrak{C}_S$ and $M \neq N/2$,*

$$|\mathcal{A}^*(M)| \leq |\mathcal{A}_{\mathcal{C}}(M)|.$$

Proof. Fix $M \in [N]$, $M \neq N/2$. By an argument similar to the proof of the converse part of Lemma S.3.4, there is an interval $[a_M, b_M]$ such that $b_M - a_M + 1 = |\mathcal{A}_{\mathcal{C}}(M)|$ and $P_M([a_M, b_M]) \geq P_M(\mathcal{A}_{\mathcal{C}}(M))$. Since $M \neq N/2$, $\mathcal{A}^*(M) = [a_M^*, b_M^*]$ is size-optimal by Theorem 3.1 so

$$|\mathcal{A}^*(M)| = b_M^* - a_M^* + 1 \leq b_M - a_M + 1 = |\mathcal{A}_{\mathcal{C}}(M)|. \tag{S.24}$$

□

Lemma S.6.2. *For even N, assume $A \subseteq [n]$ is nonempty and such that $x \in A \Rightarrow n - x \in A$. Then there exists $c \in [n]$ such that*

$$P_{N/2}([c, n-c]) \geq P_{N/2}(A) \tag{S.25}$$

and

$$n - 2c + 1 = \begin{cases} |A|, & \text{if } n \text{ or } |A| \text{ is odd,} \\ |A| + 1, & \text{if } n \text{ and } |A| \text{ are even.} \end{cases}$$

Proof. If n is odd then $x \neq n - x$ for all $x \in [n]$, implying that $|A|$ is even. Let $c = (n - |A| + 1)/2$, an integer. If $|A| = 0$ then there is nothing to prove for (S.25), so assume $|A| \geq 2$, whence $c \leq (n-1)/2 =: m$, which by (S.2) is the mode of the Hyper$(N/2, n, N)$ density. Thus $m \in [c, n-c]$ and by (S.1) this density takes the same value at the endpoints c and $n - c$. Combining these facts implies that $P_{N/2}(x_1) \geq P_{N/2}(x_2)$ for any $x_1 \in [c, n-c]$ and $x_2 \notin [c, n-c]$. Then $P_{N/2}([c, n-c]) \geq P_{N/2}(A)$ now follows from this and the fact that these two sets have the same number of points, $n - 2c + 1 = |A|$.

If n is even and $|A|$ is odd, then $c = (n - |A| + 1)/2$ is still an integer. If $|A| = 1$ then $[c, n-c] = \{n/2\}$, the point maximizing $P_{N/2}(\cdot)$ by Lemma S.1.1, hence (S.25) holds. Otherwise, $|A| \geq 3$ and $c < m$ so the argument in the previous paragraph applies.

If n and $|A|$ are both even, let $c = (n-|A|)/2$, an integer, and $B = [c, n-c-1]$. By unimodality and symmetry of $P_{N/2}(\cdot)$ about $n/2$ we have that

$$\min_{x \in B} P_{N/2}(x) = P_{N/2}(c) = P_{N/2}(n-c) = \max_{x \notin B} P_{N/2}(x).$$

It follows from this and $|B| = n - 2c = |A|$ that $P_{N/2}(B) \geq P_{N/2}(A)$, thus $P_{N/2}([c, n-c]) \geq P_{N/2}(B) \geq P_{N/2}(A)$. □

Lemma S.6.3. *In the setting of Theorem 4.1, suppose n and N are even. Then, for $\mathcal{C} \in \mathfrak{C}_S$,*

$$|\mathcal{C}^*| \leq \begin{cases} |\mathcal{C}|, & \text{if } |\mathcal{A}_{\mathcal{C}}(N/2)| \text{ is odd,} \\ |\mathcal{C}| + 1, & \text{if } |\mathcal{A}_{\mathcal{C}}(N/2)| \text{ is even.} \end{cases}$$

Proof. We have $|\mathcal{A}^*(M)| \leq |\mathcal{A}_{\mathcal{C}}(M)|$ for all $M \neq N/2$ by Lemma S.6.1. $\mathcal{A}_{\mathcal{C}}(N/2)$ is symmetrical, so by Lemma S.6.2 there is an interval $[a, n-a]$ such that $P_{N/2}([a, n-a]) \geq P_{N/2}(\mathcal{A}_{\mathcal{C}}(N/2))$ and

$$n - 2a + 1 \leq \begin{cases} |\mathcal{A}_{\mathcal{C}}(N/2)|, & \text{if } |\mathcal{A}_{\mathcal{C}}(N/2)| \text{ is odd,} \\ |\mathcal{A}_{\mathcal{C}}(N/2)| + 1, & \text{if } |\mathcal{A}_{\mathcal{C}}(N/2)| \text{ is even.} \end{cases} \tag{S.26}$$

Since $\mathcal{A}^*(N/2) = [a^*_{N/2}, b^*_{N/2}] = [a^*_{N/2}, n - a^*_{N/2}]$ is the shortest symmetrical acceptance interval for $M = N/2$, we have

$$|\mathcal{A}^*(N/2)| = b^*_{N/2} - a^*_{N/2} + 1 \leq n - 2a + 1,$$

which is thus $\leq$ the right-hand-side of (S.26). This, with the above inequality for the $M \neq N/2$ cases, gives the desired result after summing in an argument like (17). □

S.7 Supplementary material for Example 5.1

Figure S.1 is a plot of the $\mathcal{C}^*$ intervals as vertical bars for the $n = 100$ case of Example 5.1. Tables S.1-S.2 show the intervals produced by $\mathcal{C}^*$ and $\mathcal{C}_W$ for this case, and their computation times, and Figure S.2 shows the coverage probability of $\mathcal{C}_W$.

References

Johnson, N. L., Kotz, S., and Kemp, A. W. (1993). *Univariate Discrete Distributions*. John Wiley & Sons, New York, second edition.

Wang, W. (2015). Exact optimal confidence intervals for hypergeometric parameters. *Journal of the American Statistical Association*, 110(512):1491–1499.

Figure S.1: Confidence intervals $\mathcal{C}^*(x)$ for $\alpha = 0.05$, $N = 500$, $n = 100$, and $x = 0, 1, \ldots, 100$.

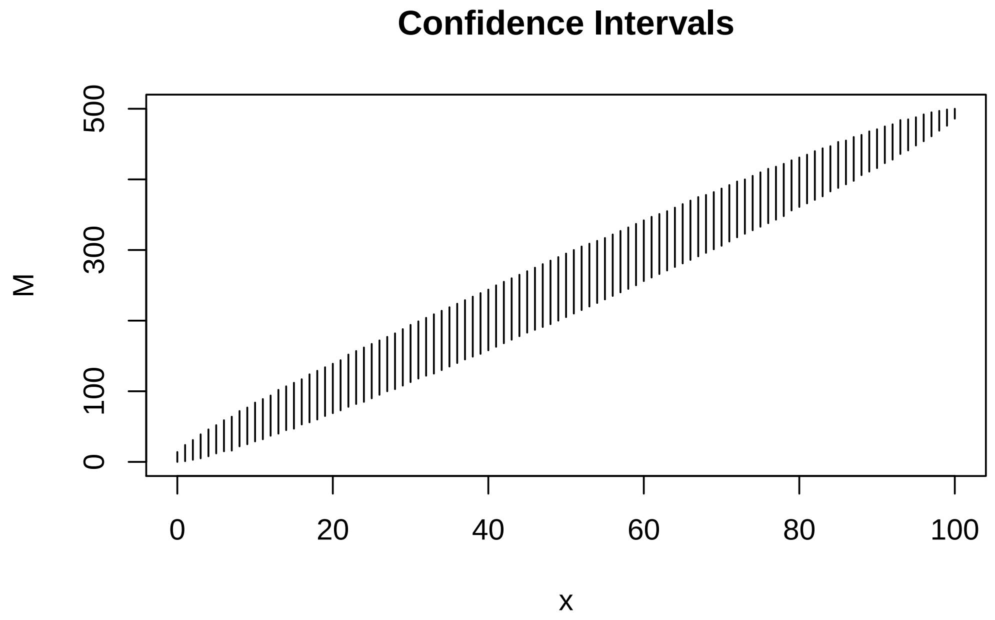

Figure S.2: Coverage probability of $\mathcal{C}_W$ for $N = 500$, $n = 100$, and $\alpha = 0.05$.

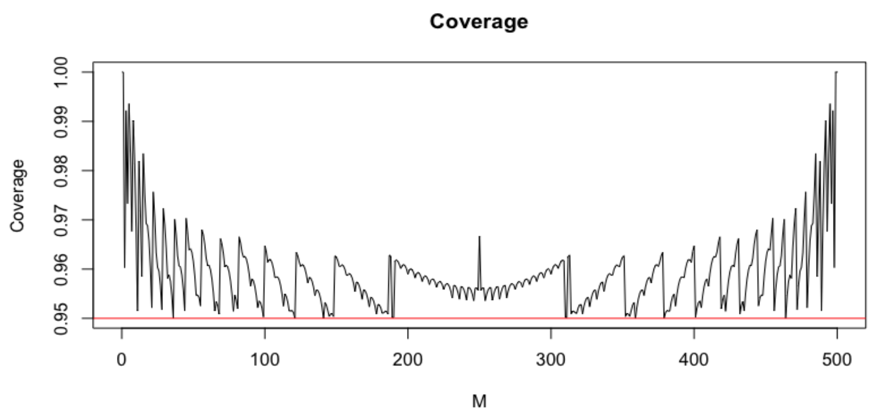

Table S.1: Confidence intervals given by $\mathcal{C}^*(x) = [L(x), U(x)]$ for $\alpha = 0.05$, $N = 500$, $n = 100$, and $x = 0, 1, \ldots, 100$.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$L(x)$	0	1	3	5	8	12	15	16	22	25	29	32	37	40
$U(x)$	14	24	31	39	46	52	59	64	72	77	84	89	94	102
x	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$L(x)$	45	47	53	56	60	65	69	73	78	82	85	90	95	100
$U(x)$	107	112	117	124	129	134	139	144	152	157	162	167	172	177
x	28	29	30	31	32	33	34	35	36	37	38	39	40	41
$L(x)$	103	108	113	118	122	125	130	135	140	145	149	153	158	163
$U(x)$	182	188	194	199	204	209	214	219	224	229	234	239	244	250
x	42	43	44	45	46	47	48	49	50	51	52	53	54	55
$L(x)$	168	173	178	183	187	191	195	200	205	210	215	220	225	230
$U(x)$	255	260	265	270	275	280	285	290	295	300	305	309	313	317
x	56	57	58	59	60	61	62	63	64	65	66	67	68	69
$L(x)$	235	240	245	250	256	261	266	271	276	281	286	291	296	301
$U(x)$	322	327	332	337	342	347	351	355	360	365	370	375	378	382
x	70	71	72	73	74	75	76	77	78	79	80	81	82	83
$L(x)$	306	312	318	323	328	333	338	343	348	356	361	366	371	376
$U(x)$	387	392	397	400	405	410	415	418	422	427	431	435	440	444
x	84	85	86	87	88	89	90	91	92	93	94	95	96	97
$L(x)$	383	388	393	398	406	411	416	423	428	436	441	448	454	461
$U(x)$	447	453	455	460	463	468	471	475	478	484	485	488	492	495
x	98	99	100											
$L(x)$	469	476	486											
$U(x)$	497	499	500											

Computational time: 0.0019 min. Total size: 7129

Table S.2: Confidence intervals given by W. Wang's (2015) method $\mathcal{C}$ for $N = 500$, $n = 100$, and $\alpha = 0.05$.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$L(x)$	0	1	3	5	8	12	15	17	22	25	29	33	37	40
$U(x)$	16	24	32	39	46	52	59	65	72	78	84	90	95	101
x	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$L(x)$	45	47	53	56	60	66	69	73	79	82	85	91	96	100
$U(x)$	107	113	117	124	130	135	141	144	152	157	163	168	173	178
x	28	29	30	31	32	33	34	35	36	37	38	39	40	41
$L(x)$	102	108	114	118	122	125	131	136	142	145	149	153	158	164
$U(x)$	182	188	194	200	205	210	215	220	225	231	236	241	246	250
x	42	43	44	45	46	47	48	49	50	51	52	53	54	55
$L(x)$	169	174	179	183	187	191	195	201	206	211	216	221	226	232
$U(x)$	253	258	263	268	274	279	284	289	294	299	305	309	313	317
x	56	57	58	59	60	61	62	63	64	65	66	67	68	69
$L(x)$	237	242	247	250	254	259	264	269	275	280	285	290	295	300
$U(x)$	321	326	331	336	342	347	351	355	358	364	369	375	378	382
x	70	71	72	73	74	75	76	77	78	79	80	81	82	83
$L(x)$	306	312	318	322	327	332	337	343	348	356	359	365	370	376
$U(x)$	386	392	398	400	404	409	415	418	421	427	431	434	440	444
x	84	85	86	87	88	89	90	91	92	93	94	95	96	97
$L(x)$	383	387	393	399	405	410	416	422	428	435	441	448	454	461
$U(x)$	447	453	455	460	463	467	471	475	478	483	485	488	492	495
x	98	99	100											
$L(x)$	468	476	484											
$U(x)$	497	499	500											

Computational time: 10.1792 min. Total size: 7131