MULTIPLE HYPOTHESIS TESTS CONTROLLING GENERALIZED ERROR RATES FOR SEQUENTIAL DATA

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Supplementary Material

This document contains proofs and auxiliary results for the paper "Multiple Hypothesis Tests Controlling Generalized Error Rates for Sequential Data".

S1 Proofs and Auxiliary Results

The proofs of the error-control properties of both the stepup and stepdown procedures utilize the following, as well as Lemma 1 that follows. Let

$$W(j,b) = \left\{ \widetilde{\Lambda}^{(j)}(n) \geq b \text{ for some } n, \widetilde{\Lambda}^{(j)}(n') > a_1 \text{ for all } n' < n \right\}, \quad \text{(S1.1)}$$

$$V_\theta(t,b) = \bigcup_{j_1,\ldots,j_t \in \mathcal{T}(\theta)} \bigcap_{\ell=1}^{t} W(j_\ell, b), \quad \text{(S1.2)}$$

$$p^{(j)}(b) = \sup_{\theta^{(j)} \in H^{(j)}} P_{\theta^{(j)}}(W(j,b)), \quad \text{(S1.3)}$$

$$M_\theta(b) = \sum_{j \in \mathcal{T}(\theta)} \mathbf{1}_{W(j,b)}. \quad \text{(S1.4)}$$

The union in (S1.2) is over all distinct t-tuples $j_1, \ldots, j_t \in \mathcal{T}(\theta)$. The event $W(j,b)$ is that the standardized test statistic associated with the jth null hypothesis crosses b from below before crossing a_1 from above, and $V_\theta(t, b)$ is the event that there are at least t true null hypotheses for which this occurs. The function $p^{(j)}(b)$ is the "worst-case" (with respect to the null) probability of $W(j, b)$ happening, and the random variable $M_\theta(b)$ is the number of true null hypotheses for which this occurs. Note that any test statistic satisfying the assumptions in Section 2.2, in particular (2.12), satisfies $p^{(j)}(b_w) \leq \alpha_w$ for all $j, w \in [J]$. Note also that the events $W(j, \cdot)$ are non-increasing in the sense that, for any $j \in [J]$,

$$b \leq b' \quad \text{implies} \quad W(j, b') \subseteq W(j, b).$$

It follows from this property that the events $V_\theta(t, \cdot)$ are non-increasing, and that $M_\theta(\cdot)$ is non-increasing with probability 1. It can also be easily verified that $V_\theta(\cdot, b)$ are non-increasing. In what follows we will frequently drop the θ from V_θ, M_θ, and other quantities when it causes no confusion.

The following lemma is an extension to the sequential domain of Lehmann and Romano (2005, Lemma 3.1).

Lemma 1. *In the testing situation above, fix $\theta \in \Theta$ such that $\mathcal{T}(\theta)$ is nonempty, let $t = |\mathcal{T}(\theta)| \in [J]$, $t_0 \in [t]$, and let $0 = \zeta_0 \leq \ldots \leq \zeta_{t_0} \leq 1$ and $b'_1 \geq b'_2 \geq \ldots \geq b'_{t_0}$ be any sequences. With $p^{(j)}(b)$ defined by* (S1.3), *if the test statistics $\{\Lambda^{(j)}(n)\}$ satisfy $p^{(j)}(b'_s) \leq \zeta_s$ for all $j \in \mathcal{T}(\theta), s \in [t_0]$, then* *with* $V_\theta(t,b)$ *defined by* (S1.2), *we have*

$$P_\theta\left(\bigcup_{s=1}^{t_0} V_\theta(s,b'_s)\right) \le t\sum_{s=1}^{t_0}\frac{\zeta_s-\zeta_{s-1}}{s}. \tag{S1.5}$$

Proof of Lemma 1. Omit θ from the notation. With $M(b)$ as in (S1.4),

$$E(M(b'_s)) = \sum_{j\in\mathcal{T}} P(W(j,b'_s)) \le \sum_{j\in\mathcal{T}} p^{(j)}(b'_s) \le t\zeta_s. \tag{S1.6}$$

Define the random variable

$$\tau = \begin{cases} \min\{s\in[t_0] : \mathbf{1}_{V(s,b'_s)} = 1\}, & \text{if } \mathbf{1}_{V(s,b'_s)} = 1 \text{ for some } s\in[t_0] \\ t+1, & \text{otherwise} \end{cases},$$

and let $\pi_s = P(\tau = s)$. Then the left-hand side of (S1.5) is

$$P\left(\bigcup_{s=1}^{t_0}\{\tau = s\}\right) = \sum_{s=1}^{t_0}\pi_s$$

by disjointness. For any $t_1\in[t_0]$ we have $\sum_{s=1}^{t_1}\tau\mathbf{1}_{\{\tau=s\}} = \tau\mathbf{1}_{\{\tau\le t_1\}} \le M(b'_{t_1})$ by definition of V and W. Taking expectations and using (S1.6) gives $\sum_{s=1}^{t_1} s\pi_s \le t\zeta_{t_1}$. Dividing both sides of this last by $t_1(t_1+1)$ (resp. t_1) for $t_1 = 1,\ldots,t_0-1$ (resp. $t_0 = t_1$) and summing over t_1 gives

$$\sum_{t_1=1}^{t_0-1}\frac{1}{t_1(t_1+1)}\sum_{s=1}^{t_1} s\pi_s + \frac{1}{t_0}\sum_{s=1}^{t_0} s\pi_s \le \sum_{t_1=1}^{t_0-1}\frac{t\zeta_{t_1}}{t_1(t_1+1)} + \frac{t\zeta_{t_0}}{t_0}. \tag{S1.7}$$

The right-hand side of (S1.7) is easily seen to be the right-hand side of (S1.5), while the left-hand side of (S1.7) simplifies to $\sum_{s=1}^{t_0}\pi_s$ after reversing the order of summation in the first term. □

S2 Proofs of Results for Stepup Procedures in Section 3.2

The proofs of both Theorems 3 and 4 utilize the following lemma.

Lemma 2. *For the generic sequential stepup procedure in Section 3.2.1, under $\theta \in \Theta$, for any $s \in [J]$ we have*

$$\{\textit{exactly } s \textit{ null hypotheses rejected}\} \subseteq V_\theta(t^*, b_s), \tag{S2.1}$$

the latter defined as

$$V_\theta(t^*, b_s) = \bigcup_t (V_\theta(t, b_s) \cap \{t^* = t\}) \tag{S2.2}$$

where $V_\theta(t, b)$ is as in (S1.2) *and t^* is the number of true null hypotheses rejected.*

Proof. Let $R(s)$ denote the event on the left-hand side of (S2.1). On outcomes in $R(s)$ define the following random variables: Let i^* be the stage at which the sth rejection occurs, let j^* be such that $H^{(j^*)}$ is the sth rejected null hypothesis, and recall that t^* is the number of true hypotheses rejected. By definition of $H^{(j^*)}$ and by step 2(a) of the procedure we have $j^* = j(n_{i^*}, |\mathcal{J}_{i^*}| - m_{i^*} + 1)$, $s = r_{i^*} + m_{i^*}$, and $\widetilde{\Lambda}^{(j^*)}(n_{i^*}) \geq b_{r_{i^*}+m_{i^*}}$. If a true hypothesis $H^{(j')}$, $j' \in \mathcal{T}$, is rejected then it is rejected at some stage $i' \leq i^*$, and $j' = j(n_{i'}, \ell)$ for some $\ell \geq |\mathcal{J}_{i'}| - m_{i'} + 1$. Note that $r_{i'} + m_{i'} \leq s$

because if $i' = i^*$ then $r_{i'} + m_{i'} = r_{i^*} + m_{i^*} = s$, and otherwise $i' \leq i^* - 1$ so $r_{i'} + m_{i'} = r_{i'+1} \leq r_{i^*} \leq s$. Then

$$\widetilde{\Lambda}^{(j')}(n_{i'}) = \widetilde{\Lambda}^{(j(n_{i'},\ell))}(n_{i'}) \geq \widetilde{\Lambda}^{(j(n_{i'},|\mathcal{J}_{i'}|-m_{i'}+1))}(n_{i'}) \geq b_{r_{i'}+m_{i'}} \geq b_s,$$

using (3.23) for the second-to-last inequality. This holds for any rejected true hypothesis, hence there are distinct $j_1, \ldots, j_{t^*} \in \mathcal{T}$ such that, for each $\ell \in [t^*]$, $H^{(j_\ell)}$ is rejected and

$$\widetilde{\Lambda}^{(j_\ell)}(n) \geq b_s \quad \text{for some } n. \tag{S2.3}$$

If it were that $\widetilde{\Lambda}_{n'}^{(j_\ell)} \leq a_1$ for some n' less than the corresponding n in (S2.3), then $H^{(j_\ell)}$ would not have been rejected but rather accepted, contradicting our assumption about $H^{(j_\ell)}$. Combining these statements gives that any outcome in $R(s)$ is in $V(t^*, b_s)$. $\square$

Proof of Theorem 3. We consider the generic stepup procedure defined in Section 3.2.1 with step values given by (3.24). Fix $\theta \in \Theta$ such that $\mathcal{T}(\theta)$ is nonempty, and omit θ from the notation. We will show that γ_1-FDP $\leq \alpha$, the other claim being similar. For $s \in [J]$ let $\gamma(s) = \lfloor \gamma_1 s \rfloor + 1$ and let $T(s)$ denote the event that at least $\gamma(s)$ true null hypotheses are rejected, and let $R(s)$ be the event on the left-hand side of (S2.1). We claim that

$$R(s) \cap T(s) \subseteq \begin{cases} V(\gamma(s) \vee (s+|\mathcal{T}|-J), b_s), & \text{if } \gamma(s) \leq |\mathcal{T}|, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{S2.4}$$

By Lemma 2 we have $R(s) \cap T(s) \subseteq V(t^*, b_s) \cap T(s)$, and to finish the proof of (S2.4) we show that, on any outcome in the latter event, defined analogously to (S2.2), $t^* \geq \gamma(s) \vee (s+|\mathcal{T}|-J)$ if s is such that $\gamma(s) \leq |\mathcal{T}|$ and then use that $V(\cdot, b)$ is non-increasing; the other case of (S2.4) is trivial by the definition of $R(s)$ and $T(s)$. We recall that this and other inequalities involving the random variable t^* should be interpreted as holding with $P(\cdot|V(t^*, b_s) \cap T(s))$-probability 1, this event assumed without loss of generality to have positive probability. We know that $t^* \geq \gamma(s)$ by definition of $T(s)$. On the other hand, t^* is equal to the number s of null hypotheses rejected minus the number of false null hypotheses rejected, the latter bounded above by $J - |\mathcal{T}|$, hence $t^* \geq s + |\mathcal{T}| - J$.

With (S2.4) established we have

$$\begin{aligned} \gamma_1\text{-FDP} = \bigcup_{1 \leq s \leq J} R(s) \cap T(s) &\subseteq \bigcup_{1 \leq s \leq J,\, \gamma(s) \leq |\mathcal{T}|} V(\gamma(s) \vee (s+|\mathcal{T}|-J), b_s) \\ &= \bigcup_{|\mathcal{T}|-J+1 \leq s \leq |\mathcal{T}|,\, \gamma(J+s-|\mathcal{T}|) \leq |\mathcal{T}|} V(\gamma(J+s-|\mathcal{T}|) \vee s, b_{J+s-|\mathcal{T}|}). \end{aligned} \tag{S2.5}$$

For s in the range of the union in (S2.5), let $\sigma(s) = \gamma(J+s-|\mathcal{T}|) \vee s$, which is a non-decreasing sequence of consecutive integers taking the values

$1, 2, \ldots, s_1$ for some $s_1 \leq |\mathcal{T}|$ by virtue of the restrictions in (S2.5). For $s \in [s_1]$ let $\sigma^{-1}(s) = \max\{s' : \sigma(s') = s\}$. If $\sigma(s) = \sigma(s+1)$ then using the non-increasing property of $V(s, \cdot)$ we have

$$V(\sigma(s), b_{J+s-|\mathcal{T}|}) \cup V(\sigma(s+1), b_{J+s+1-|\mathcal{T}|}) \subseteq V(\sigma(s+1), b_{J+s+1-|\mathcal{T}|}). \quad \text{(S2.6)}$$

By collapsing terms in (S2.5) according to (S2.6), we have that the union in (S2.5) is contained in

$$\bigcup_{s=1}^{s_1} V(s, b_{J+\sigma^{-1}(s)-|\mathcal{T}|}).$$

Denote $S_2(|\mathcal{T}|, \gamma_1, \{\delta_j\})$ and $D_2(\gamma_1, \{\delta_j\})$ by S_2 and D_2, respectively. Applying Lemma 1 to this last with $\zeta_s = \alpha_{J+\sigma^{-1}(s)-|\mathcal{T}|}$ and $b'_s = b_{J+\sigma^{-1}(s)-|\mathcal{T}|}$, and recalling that $s_1 \leq |\mathcal{T}|$, we have

$$\begin{aligned} \frac{D_2}{|\mathcal{T}|\alpha} \cdot P(\text{FDP} > \gamma_1) &\leq \frac{D_2}{|\mathcal{T}|\alpha} \cdot P\left(\bigcup_{s=1}^{s_1} V(s, b_{J+\sigma^{-1}(s)-|\mathcal{T}|})\right) \\ &\leq \frac{D_2}{|\mathcal{T}|\alpha} \cdot |\mathcal{T}| \left(\alpha_{J+\sigma^{-1}(1)-|\mathcal{T}|} + \sum_{1<s\leq s_1} \frac{\alpha_{J+\sigma^{-1}(s)-|\mathcal{T}|} - \alpha_{J+\sigma^{-1}(s-1)-|\mathcal{T}|}}{s}\right) \\ &= \delta_{J+\sigma^{-1}(1)-|\mathcal{T}|} + \sum_{1<s\leq s_1} \frac{\delta_{J+\sigma^{-1}(s)-|\mathcal{T}|} - \delta_{J+\sigma^{-1}(s-1)-|\mathcal{T}|}}{s}. \quad \text{(S2.7)} \end{aligned}$$

We claim that (S2.7) is equal to

$$S_2/|\mathcal{T}| = \delta_1 + \sum_{|\mathcal{T}|-J+1<s\leq|\mathcal{T}|,\ |\mathcal{T}|\geq\lfloor\gamma_1(J-|\mathcal{T}|+s)\rfloor+1} \frac{\delta_{J-|\mathcal{T}|+s} - \delta_{J-|\mathcal{T}|+s-1}}{s \vee (\lfloor\gamma_1(J-|\mathcal{T}|+s)\rfloor+1)}, \quad \text{(S2.8)}$$

which would complete the proof since $S_2 \leq D_2$. Note that the denominator in (S2.8) is $\sigma(s)$. If $\sigma^{-1}(1) = |\mathcal{T}| - J + 1$ then the first term in both (S2.7) and (S2.8) is δ_1. Otherwise, $\sigma^{-1}(1) = s_2 > |\mathcal{T}| - J + 1$, and the first $J + s_2 - |\mathcal{T}|$ summands in (S2.8) are

$$\delta_1 + \frac{\delta_2 - \delta_1}{\sigma(|\mathcal{T}| - J + 2)} + \ldots + \frac{\delta_{J+s_2-|\mathcal{T}|} - \delta_{J+s_2-|\mathcal{T}|-1}}{\sigma(s_2)}$$
$$= \delta_1 + \frac{\delta_2 - \delta_1}{1} + \ldots + \frac{\delta_{J+s_2-|\mathcal{T}|} - \delta_{J+s_2-|\mathcal{T}|-1}}{1} = \delta_{J+s_2-|\mathcal{T}|},$$

which is the first summand in (S2.7) in this case. Proceeding in this way one may verify the claim term by term, completing the proof. □

Proof of Theorem 4. We consider the generic stepup procedure defined in Section 3.2.1 with step values given by (3.25). We will show that the procedure satisfies k_1-FWER$_1(\theta) \leq \alpha$, the other claim being similar. Fix $\theta \in \Theta$ such that $|\mathcal{T}(\theta)| \geq k_1$, since otherwise k_1-FWER$_1(\theta) = 0$, and omit θ from the notation. For $s \in [J]$ let $T(s)$ denote the event that at least k_1 true null hypotheses are rejected and let $R(s)$ be the event on the left-hand side of (S2.1). We claim that

$$R(s) \cap T(s) \subseteq V(k_1 \vee (s + |\mathcal{T}| - J), b_s) \quad \text{for all} \quad s \in [J]. \tag{S2.9}$$

By Lemma 2 we have $R(s) \cap T(s) \subseteq V(t^*, b_s) \cap T(s)$, the latter defined analogously to (S2.2), and to finish the proof of (S2.9) we show that, on any outcome in the latter event, $t^* \geq k_1 \vee (s + |\mathcal{T}| - J)$ and use that $V(\cdot, b)$ is non-increasing. We recall that this and other inequalities involving the random variable t^* should be interpreted as holding with $P(\cdot|V(t^*, b_s) \cap T(s))$-probability 1, this event assumed without loss of generality to have positive probability. We know that $t^* \geq k_1$ by definition of $T(s)$. On the other hand, t^* is equal to the number s of null hypotheses rejected minus the number of false null hypotheses rejected, the latter bounded above by $J - |\mathcal{T}|$, hence $t^* \geq s + |\mathcal{T}| - J$.

With (S2.9) established we have

$$\bigcup_{k_1 \leq s \leq J} R(s) \cap T(s) \subseteq \bigcup_{k_1 \leq s \leq J} V(k_1 \vee (s + |\mathcal{T}| - J), b_s) \tag{S2.10}$$

$$= \left\{ \bigcup_{k_1 \leq s \leq J - |\mathcal{T}| + k_1} V(k_1, b_s) \right\} \cup \left\{ \bigcup_{J - |\mathcal{T}| + k_1 < s \leq J} V(s + |\mathcal{T}| - J, b_s) \right\}$$

$$\subseteq V(k_1, b_{J - |\mathcal{T}| + k_1}) \cup \left\{ \bigcup_{J - |\mathcal{T}| + k_1 < s \leq J} V(s + |\mathcal{T}| - J, b_s) \right\} \tag{S2.11}$$

$$= \bigcup_{J - |\mathcal{T}| + k_1 \leq s \leq J} V(s + |\mathcal{T}| - J, b_s) = \bigcup_{k_1 \leq s \leq |\mathcal{T}|} V(s, b_{|\mathcal{T}| - J + s}), \tag{S2.12}$$

where the inclusion in (S2.11) follows from the facts that $b_s \geq b_{J - |\mathcal{T}| + k_1}$ for $s \leq J - |\mathcal{T}| + k_1$ and the $V(k_1, \cdot)$ are non-increasing. The k_1-FWER$_1$ is the probability of the event on the left-hand side of (S2.10), and applying Lemma 1 to the last union in (S2.12) with $t_0 = |\mathcal{T}|$, $\zeta_0 = \ldots = \zeta_{k_1 - 1} = 0$, $\zeta_s = \alpha_{|\mathcal{T}| - J + s}$ for $k_1 \leq s \leq |\mathcal{T}|$, $b'_1 = \ldots = b'_{k_1 - 1} = \infty$, and $b'_s = b_{|\mathcal{T}| - J + s}$ for

$k_1 \leq s \leq |\mathcal{T}|$, we have

$$k_1\text{-FWER}_1 \leq |\mathcal{T}| \left(\frac{\alpha_{|\mathcal{T}|-J+k_1}}{k_1} + \sum_{k_1 < s \leq |\mathcal{T}|} \frac{\alpha_{|\mathcal{T}|-J+s} - \alpha_{|\mathcal{T}|-J+s-1}}{s} \right)$$

$$= \left(\frac{\alpha}{D_3(k_1, \{\delta_j\})} \right) S_3(k_1, |\mathcal{T}|, \{\delta_j\}) \leq \alpha.$$

□

S3 Proof of Theorem 7

We verify the first parts of (5.35) and (5.36); the other parts are similar. We have

$$0 \leq \alpha_w + \widetilde{\beta}_w = \alpha_w + \frac{\beta_1(1-\alpha_w)}{1-\alpha_1} - 1 + 1 = \frac{-(1-\alpha_w)(1-\alpha_1-\beta_1)}{1-\alpha_1} + 1 \leq 1,$$

using $\alpha_1 + \beta_1 \leq 1$ for the last inequality. It is simple algebra to check that $B_w^{(j)}$ in (5.34) can be written as $B_W(\alpha_w, \widetilde{\beta}_w)$, and $A_1^{(j)}$ in (5.34) can be written as $A_W(\alpha_w, \widetilde{\beta}_w)$ for any $w \in [J]$. Then

$$\begin{aligned}
p_w^{(j)} &= P_{h^{(j)}}(\Lambda^{(j)}(n) \geq B_w^{(j)} \text{ some } n, \Lambda^{(j)}(n') > A_1^{(j)} \text{ all } n' < n) \\
&= P_{h^{(j)}}(\Lambda^{(j)}(n) \geq B_W(\alpha_w, \widetilde{\beta}_w) \text{ some } n, \Lambda^{(j)}(n') > A_W(\alpha_w, \widetilde{\beta}_w) \text{ all } n' < n) \\
&= \alpha_W^{(j)}(\alpha_w, \widetilde{\beta}_w),
\end{aligned}$$

by definition of $\alpha_W^{(j)}$. $\square$